Binary operation:

Definition: A binary operation $*$ on a set A is a function $*: A \times A \to A$. We denote $*(a,b)$ by $a * b$

NOTE: 1) The no. of binary operations on a set containing n elements is 2^{n^2}

2) Addition $(+)$ is a binary operation on N, W, Z, Q, Q^* and R

3) Subtraction $(-)$ is a binary operation on Z, Q, Q^* and R

4) Multiplication $(\times)$ is a binary operation on N, W, Z, Q, Q^* and R

5) Division $(\div)$ is a binary operation on Q, Q^* and R

Laws of Binary operation.

1) Closure law:

$\forall\ a, b \in A \Rightarrow a * b \in A$

2) Commutative law.

$\forall\ a, b \in A \Rightarrow a * b = b * a$

3) Associative law:

$\forall\ a, b, c \in A \quad a * (b * c) = (a * b) * c$

4) Existence of identity:

$\forall\ a \in A \quad \exists\ e \in A$ such that

$a * e = a = e * a$

5) Existence of Inverse:

$\forall\ a \in A\ \exists\ a^{-1} \in A$ such that

$a * a^{-1} = e = a^{-1} * a$

Composition table

Let $A = \{a, b, c, d\}$ be any set then the composition table is given by

$*$	a	b	c	d
a	$a*a$	$a*b$	$a*c$	$a*d$
b	$b*a$	$b*b$	$b*c$	$b*d$
c	$c*a$	$c*b$	$c*c$	$c*d$
d	$d*a$	$d*b$	$d*c$	$d*d$

Ex. 1.4

1. Determine whether or not each of the definition $*$ given below gives a binary operation. In the event that $*$ is not a binary operation. Give justification for this

(i) on Z^+, define $*$ by $a*b = a-b$

Ans $\forall\ a, b \in Z^+ \Rightarrow a*b = a-b \notin Z^+$

$(\because 2, 4 \in Z^+ \Rightarrow 2*4 = 2-4 = -2 \notin Z^+]$

$\therefore *$ is not a Binary operation.

(ii) on Z^+, define $*$ by $a*b = ab$

Ans $\forall\ a, b \in Z^+ \Rightarrow a*b = ab \in Z^+$

$\therefore *$ is a Binary operation.

(iii) on R. define * by $a*b = ab^2$

$\forall a, b \in R \Rightarrow a*b = ab^2 \in R$

$\therefore$ * is a binary operation.

(iv) on Z^+ define * by $a*b = |a-b|$

$\forall a, b \in R \Rightarrow a*b = |a-b| \in R$

$\therefore$ * is a binary operation.

(v) on Z^+ define * by $a*b = a$

$\forall a, b \in Z^+ \Rightarrow a*b = a \in Z^+$

$\therefore$ * is a binary operation.

(vi) On R define * by $a*b = a+4b^2$

$\forall a, b \in R \Rightarrow a*b = a+4b^2 \in R$

$\therefore$ * is a binary operation.

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2) For each operation * defined below determine whether * is binary, commutative or associative.

(i) on Z define $a*b = a-b$

$b*a = b-a$

Clearly $a*b \neq b*a$

$\therefore$ * is not commutative.

Associative law:

$\forall a, b, c \in Z$

$a*(b*c) = a*(b-c) = a-(b-c) = a-b+c$

$(a*b)*c = (a-b)*c = (a-b)-c = a-b-c$

clearly $a*(b*c) \neq (a*b)*c$

$\therefore$ * is not Associative.

(ii) On Q define $a*b = ab+1$

Ans- Given that $a*b = ab+1 \quad \forall\ a,b \in Q$

Commutative law! $\forall\ a,b \in Q$

$$a*b = ab+1$$
$$b*a = ba+1 = ab+1$$

clearly $a*b = b*a$

$\therefore$ $*$ is Commutative.

Associative law; $\forall\ a,b,c \in Q$

$$a*(b*c) = a*(bc+1) = a(bc+1)+1$$
$$= abc+a+1$$

$$(a*b)*c = (ab+1)*c = (ab+1)c+1$$
$$= abc+c+1$$

$\therefore$ $*$ is not associative.

(iii) on Q define $a*b = \frac{ab}{2}$

Ans Given that $a*b = \frac{ab}{2} \quad \forall\ a,b \in Q$

commutative law: $\forall\ a,b \in Q$

$$a*b = \frac{ab}{2}$$
$$b*a = \frac{ba}{2} = \frac{ab}{2}$$

Here $a*b = b*a$

$\therefore$ $*$ is Commutative

Associative law! $\forall$ ~~a,b~~ a,b,c $\in Q$

$$a*(b*c) = a*\left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

$$(a*b)*c = \left(\frac{ab}{2}\right)*c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

$\therefore *$ is associative.

(iv). on Z^+ define $a*b = 2^{ab}$

Given that $a*b = 2^{ab}$ $\forall\, a,b \in Z^+$

Commutative law: $\forall\, a,b \in Z^+$

$a*b = 2^{ab}$

$b*a = 2^{ba} = 2^{ab}$

clearly $a*b = b*a$

$\therefore *$ is Commutative.

Associative law: $\forall\, a,b,c \in Z$

$a*(b*c) = a*(2^{bc}) = 2^{a \cdot 2^{bc}}$

$(a*b)*c = (2^{ab}*c) = 2^{2^{ab} \cdot c}$

$\therefore a*(b*c) \neq (a*b)*c$

V on Z^+; define $a*b = a^b$

Commutative law: $\forall\, a,b \in Z^+$

$a*b = a^b$

$b*a = b^a$

$\therefore$ ~~$a*b \neq b*a$~~ $a*b \neq b*a$

$\therefore *$ is not Commutative.

Associative law: $\forall\, a,b \in Z^+$

$(a*b)*c = a^b*c = (a^b)^c = a^{bc}$

$a*(b*c) = a*b^c = a^{b^c}$

$\therefore (a*b)*c \neq a*(b*c)$

(vi) on $R-\{-1\}$ define $a*b = \frac{a}{b+1}$

Commutative: $\forall\ a,b \in R-\{-1\}$

$$a*b = \frac{a}{b+1}$$

$$b*a = \frac{b}{a+1}$$

clearly $a*b \neq b*a$

Associative law: $\forall\ a,b,c \in R-\{-1\}$

$$a*(b*c) = a*\left(\frac{b}{c+1}\right) = \frac{a}{\frac{b}{c+1}+1} = \frac{a}{\frac{b+c+1}{c+1}}$$

$$= \frac{(c+1)a}{b+c+1}$$

$$(a*b)*c = \frac{a}{b+1}*c = \frac{\frac{a}{b+1}}{c+1}$$

$$= \frac{a}{(b+1)(c+1)}$$

$\therefore\ a*(b*c) \neq (a*b)*c$

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3) Consider the binary operation on the set $\{1,2,3,4,5\}$ defined by $a \wedge b = \min\{a$ write the operation table of the operation $\wedge$.

Ans By data $\wedge$ is a binary operation on $\{1,2,3,4,5\}$ defined by

$$a \wedge b = \min\{a,b\}$$

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

4) Consider a binary operation $*$ on the set $\{1, 2, 3, 4, 5\}$ given by following multiplication table
(i) Compute $(2*3)*4$ and $2*(3*4)$
(ii) Is $*$ Commutative
(iii) Compute $(2*3)*(4*5)$

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Ans i) $(2*3)*4 = 1*4 = 1$
$2*(3*4) = 2*1 = 1$

(ii) yes $*$ is Commutative.

(iii) $(2*3)*(4*5) = 1*1 = 1$

5) Let $*'$ be the binary operation on the set {1, 2, 3, 4, 5} defined by $a*b$ = H.C.F of a and b. Is the operation $*'$ same as the operation $*$ defined in exercise 4 above. Justify your answer.

Ans: $a, b \in \{1, 2, 3, 4, 5\}$

$a*b$ = H.C.F of a and b

composition table for $*'$

$*'$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Both the composition tables are exactly same. hence the operation $*$ and $*'$ are same.

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6) Let $*$ be the binary operation on N given by $a*b$ = L.C.M of a and b

Find (i) $5*7$, $20*16$

(ii) Is $*$ commutative

(iii) Is $*$ associative

iv) Find the identity of $*$ in N

(v) which elements of N are invertible for the operation $*$?

~~Ans~~. (i) $5*7$, $20*16$

Ans: $5*7 =$ L.C.M of 5 and 7 $= 35$

$20*16 =$ L.C.M of 20 and 16 $= 80$

(ii) $a*b =$ L.C.M of $(a,b) =$ L.C.M of (b,a)

$= b*a$

Hence $*$ is Commutative.

(iii) $(a*b)*c =$ L.C.M of a,b,c

$a*(b*c) =$ L.C.M of a,b,c

Hence $*$ is associative.

(iv) Let e be the identity element, then

$a*e = e*a = a \quad \forall\, a \in N$

L.C.M of $(a,e) =$ L.C.M of $(e,a) = a$

L.C.M of $(a,1)$ is 1

1 is identity element of N

(v) Let $a \in N$ $\exists\, b \in N$ such that

$a*b = b*a =$ L.C.M of $(a,b) = 1$

$\Rightarrow a = 1,\ b = 1$

only invertible element in N is 1

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7) Is $*$ defined on the set $\{1, 2, 3, 4, 5\}$ by $a*b =$ ~~H.C.F~~ L.C.M of a and b a binary operation? Justify your answer.

Ans Given $A = \{1, 2, 3, 4, 5\}$

$\forall a, b \in A \Rightarrow a * b =$ L.C.M of a and $b \notin A$

Ex. $4, 5 \in A \Rightarrow$ L.C.M of 4 and 5 = 20

$20 \notin A$

Hence $*$ is not a binary operation.

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8) Let $*$ be the binary operation N defined by $a * b =$ H.C.F of a and b. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on N.?

Ans $\forall a, b \in N$ $a * b =$ H.C.F of a and b

$b * a =$ HCF of b and a

$\therefore a * b = b * a$

Hence $*$ is commutative.

$\forall a, b, c \in N$ $a * (b * c) = a * ($H.C.F of b and c$)$

$=$ H.C.F of a, b and c

$(a * b) * c =$ (H.C.F of a and b) $* c$

$=$ H.C.F of a, b and c

$\therefore a * (b * c) = (a * b) * c$

$\therefore *$ is Associative

Let $e \in N$ be the identity element

$a * e = e * a = a$

$\Rightarrow$ H.C.F of a and $e = a$

$\therefore$ a divides e $\forall a \in N$

Which is not possible as there does not exist such a number e, which is divisible by every number.

$\therefore$ identity element does not exist

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9) Let $*$ be a binary operation on the set Q of rational numbers as follows.

(i) $a*b = a-b$ (ii) $a*b = a^2+b^2$

(iii) $a*b = a+ab$ (iv) $a*b = (a-b)^2$

(v) $a*b = \frac{ab}{4}$ (vi) $a*b = ab^2$

Find which of the binary operation. Commutative or Associative.

Ans i) Given $a*b = a-b$ $\forall a, b \in Q$

$b*a = b-a$

$\therefore$ $a-b \neq b-a \Rightarrow a*b \neq b*a$

Hence $*$ is not Commutative

$\forall a, b, c \in Q$

$a*(b*c) = a*(b-c) = a-(b-c)$

$= a-b+c$

$(a*b)*c = (a-b)*c = a-b-c$

$\therefore$ $a*(b*c) \neq (a*b)*c$

$\therefore$ $*$ is not Associative.

(ii) $a*b = a^2+b^2$

Ans Given $a*b = a^2+b^2 \quad \forall\, a,b \in Q$

$b*a = b^2+a^2 = a^2+b^2$

$\therefore a*b = b*a$

$\therefore *$ is Commutative.

$\forall\, a,b \in Q \quad a*(b*c) = a*(b^2+c^2)$

$= a^2+(b^2+c^2)^2$

$= a^2+b^4+c^4+2b^2c^2$.

$(a*b)*c = (a^2+b^2)*c$

$= (a^2+b^2)^2+c^2$

$= a^4+b^4+2a^2b^2+c^2$

Here $a*(b*c) \neq (a*b)*c$

$\therefore *$ is not associative.

(iii) $a*b = a+ab$

Ans $\forall\, a,b \in Q$, $a*b = a+ab$

$b*a = b+ba = b+ab$

$\therefore a*b \neq b*a$

$\therefore *$ is not Commutative.

$\forall\, a,b,c \in Q$

$a*(b*c) = a*(b+bc)$

$= a+a(b+bc)$

$= a+ab+abc$

$(a*b)*c = (a+ab)*c$

$= a+ab+(a+ab)c$

$= a+ab+ac+abc$

$\therefore \; a*(b*c) = (a*b)*c$

$\therefore$ $*$ is not Associative.

(iv) $a*b = (a-b)^2$

Ans. $\forall\, a, b \in Q$ $\quad a*b = (a-b)^2$

$b*a = (b-a)^2$

Clearly $a*b = b*a$

$\therefore$ $*$ is Commutative.

$\forall\, a, b, c\, Q \quad a*(b*c) = a*(b-c)^2$

$= (a-(b-c)^2)^2$

$= a^2 - 2a(b-c)^2 + (b-c)^4$

$(a*b)*c = (a-b)^2 * c$

$= ((a-b)^2 - c)^2$

$= (a-b)^4 + c^2 - 2(a-b)^2 c$

$\therefore \; a*(b*c) = (a*b)*c$

$\therefore$ $*$ is not Associative.

(v) $a*b = \frac{ab}{4}$

Ans $\forall\, a, b \in Q$

$a*b = \frac{ab}{4}$, $\quad b*a = \frac{ba}{4} = \frac{ab}{4}$

$\therefore a*b = b*a$

$\therefore *$ is Commutative.

$\forall a, b, c \in Q$, $a*(b*c) = a\left(\frac{bc}{4}\right) / 4 = \frac{abc}{16}$

$(a*b)*c = \frac{ab}{4}*c = \frac{abc}{16}$

$\therefore a*(b*c) = (a*b)*c$

$\therefore *$ is associative.

vi) $a*b = ab^2$

Ans $a*b = ab^2 \quad \forall a, b \in Q$

$b*a = ba^2$

$\therefore a*b \neq b*a$

$*$ is not Commutative.

$\forall a, b, c \in Q \quad a*(b*c) = a*(bc^2)$

$= a(bc^2)^2 = ab^2c^4$

$(a*b)*c = (ab^2)*c$

$= ab^2c^2$

$\therefore a*(b*c) \neq (a*b)*c$

Find which of the operations given above has identity.

i) $a*b = a-b$

Ans. $\forall a \in Q \ \exists e \in Q$ such that

$a*e = a \quad e*a = a$

$a*e = a-e$

$e*a = e-a \quad$ clearly $a*e \neq e*a$

$\therefore$ Identity does not exist

(ii) $a * b = a^2 + b^2$

$\forall a \in Q \; \exists \; e \in Q$ such that $a * e = e * a$

$a * e = a^2 + e^2 = a$

$e * a = e^2 + a^2 = a$

which is not possible as $a^2 \neq a$

$\forall a \in Q$

hence identity element does not exist.

(iii) $a * b = a + ab$

$\forall a \in Q \; \exists \; e \in Q$ such that

$a * e = a$ $\qquad$ $e * a = a$

$a + ae = a$ $\qquad$ $e + ae = a$

$ae = 0$ $\qquad$ $e(1+a) = a$

$\Rightarrow e = 0$ $\qquad$ $e = \dfrac{a}{1+a}$

$\therefore$ identity element does not exist

(iv) $a * b = (a-b)^2$

$\forall a \in Q \; \exists \; e \in Q$ such that

$a * e = a$ $\qquad$ $e * a = a$

$(a-e)^2 = a$ $\qquad$ $(e-a)^2 = a$

$a - e = \pm a$ $\qquad$ $e - a = \pm a$

$\Rightarrow e = 0$ and $e = 2a$

which is not possible

Hence ~~*~~ ~~is~~ identity element

does not exist.

V $a * b = \frac{ab}{4}$

$\forall\, a \in Q\ \exists\ e \in Q$ such that

$a * e = a \qquad e * a = a$

$\frac{ae}{4} = a \qquad \frac{ea}{4} = a$

$e = 4 \qquad e = 4$

$\therefore$ Identity element is 4

(vi) $a * b = ab^2$

$\forall\, a \in Q\ \exists\ e \in Q$ such that

$a * e = a \qquad e * a = a$

$ae^2 = a \qquad ea^2 = a$

$\therefore e^2 = 1 \qquad ea = 1$

$e = \pm 1 \qquad e = \frac{1}{a}\ \forall\, a \in Q$

Identity does not exist

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ii) Let $A = N \times N$ and $*$ be the binary operation on A defined by

$$(a,b) * (c,d) = (a+c, b+d)$$

Show that $*$ is commutative and associative. Find the identity element for $*$ on A if any.

Ans Given $(a,b) * (c,d) = (a+c, b+d)$

$\Rightarrow (a+c, d+b) = (c,d) * (a,$

$\Rightarrow$ ~~$(b,a) * (d,c)$~~

Hence $*$ is commutative

Let (e, f) be the identity element of A

$(e, f) * (a, b) = (a, b) * (e, f) = (a, b)$

$(a+e, b+f) = (a, b) \Rightarrow (e = 0, f = 0) \notin$

Hence no identity element in N

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12) State whether the following statements are true or false. Justify.

(i) For an arbitary binary operation $*$ on a set N $a * a = a \ \forall a \in N$

(ii) If $*$ is a commutative binary operation on N then $a * (b * c) = (c * b) * a$

(i) Ans. If $a * b = a + b$ then

$a * a = a + a = 2a \neq a$

$\therefore$ statement is false.

(ii) Ans Since commutative.

$a * (b * c) = (b * c) * a$

$= (c * b) * a$

Hence statement is true.

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13) Consider a binary operation $*$ on N defined as $a * b = a^3 + b^3$, Choose the correct answer.

A) Is $*$ both associative and commutative

B). Is $*$ commutative but not associative

c) Is * associative but not commutative

D) Is * neither commutative nor associative.

Ans $a * b = a^3 + b^3 = b^3 + a^3 = b * a$

$a * b = b * a$

* is commutative.

$(a * b) * c = (a^3 + b^3) * c = (a^3 + b^3)^3 + c^3$

$a * (b * c) = a * (b^3 + c^3) = a^3 + (b^3 + c^3)^3$

Hence * is not associative.

'B' is correct answer.