Functions

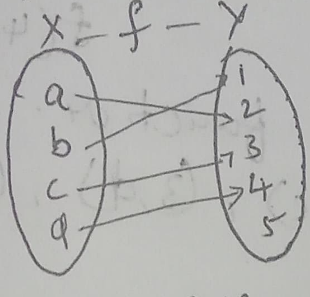
Types of functions:

1) One-One (injective function) function:

A function $f: X \to Y$ is difined to be one-one (or injective) if the images of distinct elements of X and under f are distinct: i.e for every $x_1, x_2 \in X$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Ex.



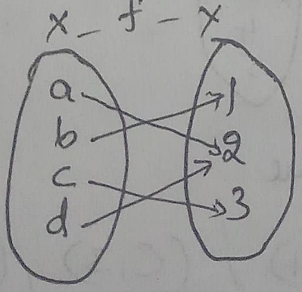
one-one

$f(a) = 2$
$f(b) = 1$
$f(c) = 3$
$f(d) = 4$

$f = \{(a,2), (b,1), (c,3), (d,4)\}$

2) onto function (or Surjective).

A function $f: X \to Y$ is said to be onto, if every element of Y is the image of some element of X under f. i.e for every $y \in Y$, there exists an element x in X such that $f(x) = y$



$f(a) = 2$
$f(b) = 1$
$f(c) = 3$
$f(d) = 2$

onto

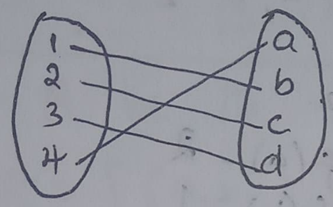
$f = \{(a,2), (b,1), (c,3), (d,2)\}$

3) <u>Bijective Function</u>:

A function $f: A \to B$ is said to be bijective (one-one and onto) if f is both one-one and onto.

Ex. If $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$

$A \xrightarrow{f} B$



ie $f(1) = b$
$f(2) = c$
$f(3) = d$
$f(4) = a$

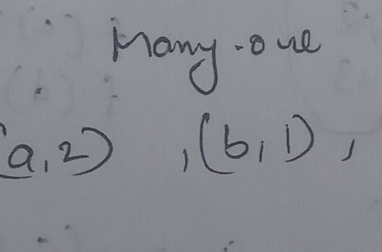
is a bijective function.

$f = \{(1, b), (2, c), (3, d), (4, a)\}$

4) <u>Many one function</u>:

A function $f: A \to B$ is said to be a many-one function if two or more elements of set A have the same images in the set B.

$A \xrightarrow{f} B$



ie $f(a) = 2$
$f(b) = 1$
$f(c) = 3$
$f(d) = 3$

Many-one

$f = \{(a, 2), (b, 1), (c, 3), (d, 3)\}$

Inverse of a function

A function $f: A \to B$ is said to be invertible if there exists a function $g: B \to A$ such that $f o g = I_B$ and $g o f = I_A$ then the function g is called the inverse of f and it is denoted by f^{-1} ie $g = f^{-1}$

NOTE

If f is invertible, then f must be one-one and onto and vice-versa

① Show that function $f: R_* \to R_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto. Where R_* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with codomain being same as R_*?

<u>One-one</u>. Let $x_1, x_2 \in R_*$, $f(x_1) = f(x_2)$

$$\frac{1}{x_1} = \frac{1}{x_2}$$

$$x_1 = x_2$$

$\therefore$ f is one-one.

<u>onto</u>: Let $x \in R_*$ $\exists$ $y \in R_*$ such that

$$f(x) = y$$

$$\frac{1}{x} = y$$

$$x = \frac{1}{y} \in R_*$$

$\therefore$ f is onto

If Domain R_* is replaced by N then

$f: N \to R_*$ defined by $f(x) = \frac{1}{x}$ $\forall x \in N$

<u>One-one</u>: Let $x_1, x_2 \in N$, $f(x_1) = f(x_2)$

$$\frac{1}{x_1} = \frac{1}{x_2}$$

$$x_1 = x_2$$

$\therefore$ f is one-one

onto : $\forall x \in N \ \exists \ y \in N$ such that $f(x)=y$

$$\frac{1}{x} = y$$

$$x = \frac{1}{y} \notin N$$

$\therefore$ f is not onto

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(2) Check the injectivity and surjectivity of the following functions:

(i) $f: N \to N$ given by $f(x) = x^2$

(ii) $f: Z \to Z$ given by $f(x) = x^2$

(iii) $f: R \to R$ given by $f(x) = x^2$

(iv) $f: N \to N$ given by $f(x) = x^3$

(v) $f: Z \to Z$ given by $f(x) = x^3$

Ans. (i) Given $f: N \to N$ defined by $f(x) = x^2$

One one : let $x_1, x_2 \in N$, $f(x_1) = f(x_2)$

$$x_1^2 = x_2^2$$

$$x_1 = x_2$$

$\therefore$ f is one-one.

onto : let $x \in N \ \exists \ y \in N$ such that

$$f(x) = y$$

$$x^2 = y$$

$x = \pm\sqrt{y} \notin N$ ($\because y = 2, x = \sqrt{2} \notin N$, $\sqrt{2}$ is irrational number)

$\therefore$ f is not onto

(ii) $f: Z \to Z$ given by $f(x) = x^2$

One-one : let $x_1, x_2 \in Z$

$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$x_1 \neq x_2 \quad (\because (-2)^2 = 2^2 \Rightarrow -2 \neq 2)$

$\therefore$ f is not one-one

Onto : let $x \in Z$ $\exists$ $y \in Z$ such that

$f(x) = y$

$x^2 = y$

$x = \pm\sqrt{y} \notin Z \quad (\because y = 2 \Rightarrow x = \sqrt{2} \notin Z)$

$\therefore$ f is not onto

(iii) $f: R \to R$ given by $f(x) = x^2$

One-One: let $x_1, x_2 \in R$, $f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$x_1 \neq x_2 \quad (\because (1)^2 = (-1)^2 \Rightarrow 1 \neq -1)$

$\therefore$ f is not one-one

Onto let $x \in R$ $\exists$ $y \in R$ such that

$f(x) = y$

$x^2 = y$

$x = \pm\sqrt{y} \notin R \quad (\because$ for $y = 2$ $x = \sqrt{2} \notin R)$

$\therefore$ f is not onto

(iv) $f: N \to N$ given by $f(x) = x^3$

<u>one-one</u>: let $x_1, x_2 \in N$

$f(x_1) = f(x_2)$

$x_1^3 = x_2^3$

$x_1 = x_2$

f is one-one.

<u>onto</u>: let $x \in N$, $\exists\, y \in N$ such that $f(x) = y$

$x^3 = y$

$x = \sqrt[3]{y} \notin N$ ($\because$ for $y = 6$, $x = \sqrt[3]{6} \notin N$)

f is not onto

V $f: Z \to Z$ given by $f(x) = x^3$

<u>one-one</u>: let $x_1, x_2 \in Z$

$f(x_1) = f(x_2)$

$x_1^3 = x_2^3$

$x_1 \neq x_2$ ($\because (-8)^3 = (8)^3 \Rightarrow$
$-8 \neq 8$)

$\therefore$ f is not one-one.

<u>onto</u>: let $x \in Z$, $\exists\, y \in Z$ such that

$f(x) = y$

$x^3 = y$

$x = \sqrt[3]{y} \notin Z$ ($\because y = 2$, $x = \sqrt[3]{2} \notin Z$

f is not onto

3) Prove that the greatest integer Function $f: R \to R$ given by $f(x) = [x]$ is neither one-one nor onto. where $[x]$ denotes the greatest integer less than or equal to x

Ans one-one: Let $x_1, x_2 \in R$, $f(x_1) = f(x_2)$

$[x_1] = [x_2]$

$x_1 \neq x_2$ $[\because [1.5] = [1.6]$

$\Rightarrow 1.5 \neq 1.6]$

$\therefore$ f is not one-one.

onto: Range of f is the set of integers

$\neq R$

hence f is not onto.

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4) Show that the Modulus function $f: R \to R$ given by $f(x) = |x|$ is neither one-one nor onto where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$ if x is negative.

Ans One-one Let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$|x_1| = |x_2|$

$x_1 \neq x_2$ $(\because |2| = |-2|$

$\Rightarrow 2 \neq -2]$

$\therefore$ f is not one-one

onto:

We know that the range of modulus function is set of all positive reals including 0 but the codomain is set of all real numbers.

$\therefore$ (Codomain) $\neq$ Range.

$\therefore$ f is not onto

Hence f is neither one-one nor onto.

5) Show that the Signum function $f: R \to R$ given by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto

Ans Given $f: R \to R$ defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \quad \forall x \in R$$

One-One let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$x_1 \neq x_2$ $\quad$ $(\because f(1) = f(2) \Rightarrow 1 \neq 2)$

$\therefore$ f is not one-one.

onto : Range of $f : \{-1, 0, 1\} \neq R$

Hence f is not onto.

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6) Let $A = \{1, 2, 3\}$ $B = \{4, 5, 6, 7\}$ and let $f = \{(1,4), (2,5), (3,6)\}$ be a function from A to B. Show that f is one-one.

Ans Given $A = \{1, 2, 3\}$ $B = \{4, 5, 6, 7\}$

$f = \{(1,4), (2,5), (3,6)\}$

$f(1) = 4$

$f(2) = 5$

$f(3) = 6$

Since images of distinct elements of A under f are distinct.

Hence It is one-one.

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7) In each of the following cases, state whether the function is one-one, onto or bijective. justify your answer.

(i) $f : R \to R$ defined by $f(x) = 3 - 4x$

(ii) $f : R \to R$ defined by $f(x) = 1 + x^2$.

Ans (i) let $x_1, x_2 \in R$

$\Rightarrow f(x_1) = f(x_2)$

$3 - 4x_1 = 3 - 4x_2$

$x_1 = x_2$

Hence f is one-one

Onto: Let $x \in R \ \exists \ y \in R$ such that

$f(x) = y$

$3 - 4x = y$

$4x = 3 - y$

$x = \frac{3-y}{4} \in R$

$\therefore$ f is onto

Hence f is bijective.

(ii) Given $f : R \to R$ defined by $f(x) = 1 + x^2$

Ans. Let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$1 + x_1^2 = 1 + x_2^2$

$x_1^2 = x_2^2$

$x_1 = \pm x_2 \Rightarrow x_1 \neq x_2 \quad (\because (-2)^2 = 2^2 \Rightarrow -2 \neq 2)$

$\therefore$ f is not one-one.

onto $\forall \ x \in R \ \exists \ y \in R$ such that $f(x) = y$

$1 + x^2 = y \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y-1} \notin R$

$(\because$ for $y = -1$ $x = \sqrt{-1-1} = \sqrt{-2} \notin R$

$\therefore$ f is not onto

Hence f is not bijective.

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8) Let A and B be sets Show that $f : A \times B \to B \times$ such that $f(a, b) = (b, a)$ is bijective function.

Ans: let $(p,q), (r,s) \in A \times B$

$f(p,q) = f(r,s)$

$\Rightarrow (q,p) = (s,r)$

$\Rightarrow q = s$ and $p = r$

$\Rightarrow (p,q) = (r,s)$

Hence f is one-one.

$\forall (b,a) \in B \times A \; \exists \; (a,b) \in A \times B$

such that $f(a,b) = f(b,a)$. hence onto

Hence f is bijective function.

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9) let $f: N \to N$ be defined by

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in N$$

State whether f is bijective?

$f(1) = \frac{1+1}{2} = \frac{2}{2} = 1$ ($\because$ n is odd)

$f(2) = \frac{2}{2} = 1$ ($\because$ n is even)

$\therefore f(1) = f(2)$

but $1 \neq 2$

Hence f is not one-one.

$\therefore$ Hence f is not bijective.

10) Let $A = R - \{3\}$ and $B = R - \{1\}$ Consider the function $f : A \to B$ defined by $f(x) = \left[\frac{x-2}{x-3}\right]$. Is f one-One and onto. Justify your answer.

Ans one-one :- Let $x_1, x_2 \in A$

$$f(x_1) = f(x_2)$$

$$\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\cancel{x_1 x_2} - 2x_2 - 3x_1 + \cancel{6} = \cancel{x_1 x_2} - 2x_1 - 3x_2 + \cancel{6}$$

$$x_1 = x_2$$

$\therefore$ f is one-one

onto : Let $y \in B$ then $f(x) = y$

$$\frac{x-2}{x-3} = y$$

$$x - 2 = y(x-3)$$

$$x - 2 = xy - 3y$$

$$x - xy = 2 - 3y$$

$$x(1-y) = 2 - 3y$$

$$x = \frac{2-3y}{1-y} \in A$$ [$\therefore \forall y \in B \ \exists x \in A$ such that $f(x) = y$]

Hence f is onto

Hence f is one-one and onto

$\therefore$ f is bijective.

11) Let $f: R \to R$ be defined as $f(x) = x^4$. Choose the correct answer.

A) f is one-one onto B) f is many-one onto C) f is one-one but not onto

D) f is neither one-one nor onto

Ans Let $x_1, x_2 \in R$, $f(x_1) = f(x_2)$

$$x_1^4 = x_2^4$$

$$x_1^4 - x_2^4 = 0$$

$$(x_1^2 - x_2^2)(x_1^2 + x_2^2) = 0$$

$$(x_1 + x_2)(x_1 - x_2)(x_1^2 + x_2^2) = 0$$

$x_1 = x_2$ or $x_1 = -x_2$

$\therefore$ f is not one-one

$f(x) = x^4 \geq 0 \quad \forall x \in R$

Range of $f = [0, \infty) \neq R$

hence not onto $\therefore$ Correct answer is D

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12) Let $f: R \to R$ be defined as $f(x) = 3x$. Choose the correct answer.

A) f is one-one onto B) f is many-one onto C) f is one-one but not onto D) f is neither one-one nor onto.

Ans Let $x_1, x_2 \in R$

$$f(x_1) = f(x_2)$$

$$3x_1 = 3x_2$$

$x_1 = x_2$ $\therefore$ f is one-one

let $y \in R$ then $\exists\, f(x) = y$

$3x = y$

$x = \frac{y}{3} \in R$.

Hence f is onto

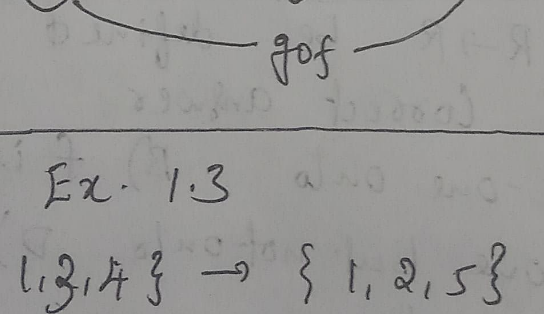
Hence f is one-one onto

$\therefore$ A is Correct answer.

Composition of two Functions

Let $f: A \to B$ and $g: B \to C$ be two function then the Composition of f and g denoted by gof and it is defined as the function $gof: A \to C$ given by

$gof(x) = g(f(x)) \quad \forall\, x \in A$



Ex. 1.3

1. Let $f: \{1, 3, 4\} \to \{1, 2, 5\}$ and $g: \{1, 2, 5\} \to \{1, 3\}$ be given by $f = \{(1,2), (3,5), (4,1)\}$ and $g = \{(1,3), (2,3), (5,1)\}$ write down gof

Ans $f = \{(1,2), (3,5), (4,1)\}$

$\Rightarrow f(1) = 2 \quad f(3) = 5 \quad f(4) = 1$

$g = \{(1,3), (2,3), (5,1)\}$

$\Rightarrow g(1) = 3 \quad g(2) = 3 \quad g(5) = 1$

$\therefore gof : \{1,3,4\} \to \{1,3\}$ defined

by. $gof(1) = g[f(1)] = g(2) = 3$

$gof(3) = g(f(3)] = g(5) = 1$

$gof(4) = g(f(4)] = g(1) = 3$

$gof = \{(1,3), (3,1), (4,3)\}$

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2) let f, g and h be functions from R to R show that

$(f+g)oh = foh + goh$

$(f \cdot g)oh = (foh) \cdot (goh)$

i) $\forall x \in R$, ~~fog~~

$[(f+g)oh](x) = [f+g](h(x))$

$= f(h(x)) + g[h(x)]$

$[(f+g)oh](x) = foh(x) + goh(x)$

$(f+g)oh = foh + goh$.

(ii) $\forall x \in R$.

$[(f \cdot g)oh](x) = (f \cdot g)[h(x)]$

$= \cancel{f[g(x)]} = f(h(x)) \times g(h(x))$

$= f[h(x)] \cdot g(h(x))$

$= foh(x) \cdot goh(x)$

$(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$

3) Find $g \circ f$ and $f \circ g$ if

(i) $f(x) = |x|$ and $g(x) = |5x-2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{1/3}$

Ans (i) $(g \circ f)(x) = g(f(x))$

$= g(|x|)$

$= |5|x|-2|$

$f \circ g(x) = f[g(x)]$

$= f(|5x-2|)$

$= ||5x-2||$

$= |5x-2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{1/3}$

$g \circ f(x) = g[f(x)]$

$= g(8x^3)$

$= (8x^3)^{1/3}$

$= (2^3 x^3)^{1/3} = [(2x)^3]^{1/3} = 2x$

$g \circ f(x) = 2x$

$f \circ g(x) = f(g(x)) = f(x^{1/3}) = 8(x^{1/3})^3 = 8x$

4) If $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$. Show that

$f \circ f(x) = x$ for all $x \neq \dfrac{2}{3}$ what is the

inverse of f.

$$f\circ f(x) = f(f(x))$$

$$= f\left[\frac{4x+3}{6x-4}\right]$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

$$= \frac{16x+\cancel{12}+18x-\cancel{12}}{\cancel{24x}+18-\cancel{24x}+16}$$

$$= \frac{34x}{34} = x$$

$\therefore$ $f\circ f(x) = x$

$\therefore$ $f = f^{-1}$ ie. Inverse of f is f itself.

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5) State with reason whether following functions have inverse.

(i) $f: \{1,2,3,4\} \to \{10\}$ with
$f = \{(1,10), (2,10), (3,10), (4,10)\}$

(ii) $g: \{5,6,7,8\} \to \{1,2,3,4\}$ with
$g = \{(5,4), (6,3), (7,4), (8,2)\}$

(iii) $h = \{2,3,4,5\} \to \{7,9,11,13\}$ with
$h = \{(2,7), (3,9), (4,11), (5,13)\}$

(i) Given $f : \{1, 2, 3, 4\} \to \{10\}$

with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

Ans. $f(1) = f(2) = f(3) = f(4) = 10$

$\Rightarrow$ f is not one - one

i.e f is many one

$\therefore$ f does not have an inverse

(ii) Given $g = \{5, 6, 7, 8\} \to \{1, 2, 3, 4\}$

$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

$g(5) = 4 \quad g(6) = 3 \quad g(7) = 4 \quad g(8) = 2$

$g(5) = g(7) = 4$

Here 5 and 7 has same images

$\therefore$ g is not one - one

Hence g does not have inverse.

(iii) $h = \{2, 3, 4, 5\} \to \{7, 9, 11, 13\}$

$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

$h(2) = 7 \quad h(3) = 9 \quad h(4) = 11 \quad h(5) = 13$

elements of h has distinct images

$\therefore$ h is one - one.

Range of $h = \{7, 9, 11, 13\}$ = Co domain

$\Rightarrow$ h is onto

Hence h has an inverse.

$h^{-1} = \{(7, 2), (9, 3), (11, 4), (13, 5)\}$

6) Sho

$f(x) =$

inverse

Ans.

$f($

One - One :

x_1

onto :

$\therefore f$

f^{-1} is

6) Show that $f : [-1, 1] \to R$ given by $f(x) = \frac{x}{x+2}$ is one-one. Find the inverse of the function $f : [-1, 1] \to$ Range f

Ans. Given $f : [-1, 1] \to R$ defined by

$$f(x) = \frac{x}{x+2} \quad \forall\ x \in [-1, 1]$$

One-one: $\forall\ x_1, x_2 \in [-1, 1]$

$$f(x_1) = f(x_2)$$

$$\frac{x_1}{x_1+2} = \frac{x_2}{x_2+2}$$

$$\cancel{x_1 x_2} + 2x_1 = \cancel{x_1 x_2} + 2x_2$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

$\therefore$ f is one-one

onto: $\forall\ x \in [-1, 1]$ $\exists$ $y \in$ Range f such that $f(x) = y$

$$\frac{x}{x+2} = y$$

$$x = xy + 2y$$

$$x(1-y) = 2y$$

$$x = \frac{2y}{1-y} \in [-1, 1]$$

$\therefore$ f is onto

$\therefore$ f is bijective and f^{-1} exist

f^{-1} is given by $f^{-1}(y) = \frac{2y}{1-y}$, $y \neq 1$

7) Consider $f: R \to R$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f

Ans: $f: R \to R$ defined by $f(x) = 4x + 3$, $\forall x \in R$.

let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + 3 = 4x_2 + 3$

$\Rightarrow x_1 = x_2$

Hence f is one-one.

let $y = 4x + 3 \Rightarrow y - 3 = 4x$

$\Rightarrow x = \frac{y-3}{4}$

$\forall y \in R \ \exists \ x \in R$ such that

$f\left(\frac{y-3}{4}\right) = \frac{4(y-3)}{4} + 3 = y$

Hence f is onto

Hence inverse exist. and

$f^{-1}: R \to R$ such that $f^{-1}(y) = \frac{y-3}{4}$

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8) Consider $f: R \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$ where R_+ is the set of all non-negative real numbers.

Ans let $x_1, x_2 \in R$

$f(x_1) = f(x_2) \Rightarrow x_1^2 + 4 = x_2^2 + 4$

$x_1^2 = x_2^2$

$\Rightarrow |x_1| = |x_2|$ as $x_1, x_2 > 0$

$\Rightarrow x_1 = x_2$

hence f is one-one.

Let $y = x^2 + 4$

$x^2 = y - 4$

$\Rightarrow x = \sqrt{y-4}$ as $x > 0$

$\forall y \in R - [4]\ \exists x \in R$ such that

$f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$

$\therefore f(\sqrt{y-4}) = y$

$\therefore f$ is invertible and

$f^{-1}(y) = \sqrt{y-4}$ $\therefore$ The inverse of f is $\sqrt{y-4}$

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9) Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6} - 1}{3}\right)$

Ans let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$\Rightarrow 9x_1^2 + 6x_1 \cancel{-5} = 9x_2^2 + 6x_2 \cancel{-5}$

$9(x_1^2 - x_2^2) = 6(x_2 - x_1)$

$9(x_1 + x_2)(x_1 - x_2) + 6(x_1 - x_2) = 0$

$\Rightarrow 3(x_1 - x_2)\{3(x_1 + x_2) + 2\} = 0$

$\Rightarrow x_1 - x_2 = 0$ as $3(x_1 + x_2) + 2 \neq 0$ and $3 \neq 0$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

let $y = 9x^2 + 6x - 5$

$y = (3x)^2 + 2(3x)1 + 1 - 1 - 5$

$y = (3x+1)^2 - 6$

$(3x+1)^2 = y + 6$

$3x + 1 = \sqrt{y+6}$

$3x = \sqrt{y+6} - 1$

$x = \dfrac{\sqrt{y+6} - 1}{3}$

$\forall\ y \in [-5, \infty)\ \exists\ x \in R_+$ such that

$f\left[\dfrac{\sqrt{y+6} - 1}{3}\right] = \left[3\left(\dfrac{\sqrt{y+6} - 1}{3}\right) + 1\right]^2 - 6 = y$

$\therefore$ f is onto

$\therefore$ f is invertible

Hence f is invertible with

$f^{-1}(y) = \left(\dfrac{\sqrt{y+6} - 1}{3}\right)$

id 3 to

10) Let $f: X \to Y$ be an invertible function. Show that f has unique inverse.
Hint: Suppose g_1 and g_2 are two inverses of f then for all $y \in Y$, $fog_1(y) = 1(y) = fog_2(y)$ use one-one ness of f)

Ans Let g_1 and g_2 are two inverses of f

then for all $y \in Y$

$$fog_1(y) = f[g_1(y)] = y = I_y(y)$$

$$fog_2(y) = f[g_2(y)] = y = I_y(y)$$

$$\therefore fog_1(y) = fog_2(y) \quad \forall y \in Y$$

$$f[g_1(y)] = f[g_2(y)] \quad \forall y \in Y$$

$$g_1(y) = g_2(y) \quad (\because f \text{ is one-one})$$

$$\therefore g_1 = g_2$$

Inverse of f is unique.

---

11) Consider $f: \{1,2,3\} \to \{a,b,c\}$ given by $f(1) = a$ $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$

Ans Given that $f = \{(1,a), (2,b), (3,c)\}$

$\therefore f$ is one-one and onto

$\therefore f$ is invertible

$f^{-1}(a) = 1$ $\quad f^{-1}(b) = 2$ $\quad f^{-1}(c) = 3$

$\therefore f^{-1}\{a,b,c\} \Rightarrow \{1,2,3\}$ is also one-one and onto

$\therefore$ Inverse of f^{-1} is $(f^{-1})^{-1}$ exist

$(f^{-1})^{-1} = \{(1,a), (2,b), (3,c)\} = f$

hence $(f^{-1})^{-1} = f$

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12) Let $f : X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f, i.e. $(f^{-1})^{-1} = f$

Ans Let $f : X \to Y$ is invertible

$\Rightarrow$ f is one-one and onto

and $f^{-1} : Y \to X$ is defined as

$f^{-1}(y) = x$

$\Rightarrow y = f(x) \quad \forall x \in X$ and $y \in Y$

Let $y_1, y_2 \in Y$

$f^{-1}(y_1) = f^{-1}(y_2)$

$f \circ f^{-1}(y_1) = f \circ f^{-1}(y_2)$

$I_y(y_1) = I_y(y_2)$

$y_1 = y_2$

$\therefore$ f^{-1} is one-one

$\forall x \in X \ \exists y \in Y$ such that

$f^{-1}(y) = x$ hence f^{-1} is onto

hence f^{-1} is invertible

ction.

let $g = (f^{-1})^{-1}$

$g o f^{-1} = I_y$ and $f o g = I_x$

$\forall \; x \in X \quad I_x(x) = x$

$(f^{-1} o g)(x) = f^{-1}[g(x)] = x$

$f o f^{-1}[g(x)] = f(x)$

$(f o f^{-1})[g(x)] = f(x)$

$g(x) = f(x)$

$g = f$

$\therefore (f^{-1})^{-1} = f$

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13) If $f : R \to R$ be given by $f(x) = (3x - x^3)^{1/3}$

then $f o f(x)$ is

A) $x^{1/3}$ B) x^3 C) $\underline{x}$ D) $3 - x^3$

Ans. $f(x) = (3 - x^3)^{1/3}$

$f o f(x) = f[(3 - x^3)^{1/3}]$

$= [3 - [(3 - x^3]^{1/3}]^3]^{1/3}$

$= (3 - 3 + x^3)^{1/3} = (x^3)^{1/3} = x$

Ans. $\underline{C}$ i.

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14) Let $f : R - \{-\frac{4}{3}\} \to R$ be a function defined as $f(x) = \frac{4x}{3x+4}$, the inverse of f is the map

g: Range $f \to R - \{-\frac{4}{3}\}$ given by

A) $g(y) = \frac{3y}{3-4y}$ B) $g(y) = \frac{4y}{4-3y}$

c) $g(y) = \frac{4y}{3-4y}$ D) $g(y) = \frac{3y}{4-3y}$

let $y = \frac{4x}{3x+4}$

$y(3x+4) = 4x$

$3xy + 4y = 4x$

$3xy - 4x = -4y$

$x(3y-4) = -4y$

$x = \frac{-4y}{3y-4} = \frac{4y}{4-3y}$

$g(y) = f^{-1}(y) = \frac{4y}{4-3y} \quad \forall\, y \in R - \boxed{\frac{4}{3}}$

Correct answer is B

<u>Extra Sums</u>

If $f: R \to R$. defined by $f(x) = 10x+7$ show that f is invertible and Find the inverse of f

let $y = f(x)$

$y = 10x+7$

$10x = y-7$

$x = \frac{y-7}{10}$

$g(x) = \frac{x-7}{10}$

Consider

$$fog(x) = f(g(x))$$
$$= f\left(\frac{x-7}{10}\right)$$
$$= \cancel{10}\left(\frac{x-7}{\cancel{10}}\right)+7 = x-7+7 = x$$

$$fog(x) = x$$

$$gof(x) = g(f(x))$$
$$= g(10x+7)$$
$$= \frac{10x+\cancel{7}-\cancel{7}}{10} = \frac{10x}{10} = x$$

$$gof(x) = x$$

$$fog(x) = gof(x)$$

$\Rightarrow$ f is invertible $\therefore f^{-1}(x) = \frac{x-7}{10}$

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2) Consider $f: R-\{2\} \to R-\{1\}$ given by $f(x) = \frac{x-3}{x-2}$ S.T f is invertible and find $f^{-1}(x)$

let $y = f(x)$

$$y = \frac{x-3}{x-2}$$

$$xy - 2y = x-3$$
$$xy - x = 2y-3$$
$$x(y-1) = 2y-3$$
$$x = \frac{2y-3}{y-1}$$

$$g(x) = \frac{2x-3}{x-1}$$

$$fog(x) = f(g(x))$$
$$= f\left(\frac{2x-3}{x-1}\right)$$
$$= \frac{\frac{2x-3}{x-1} - 3}{\frac{2x-3}{x-1} - 2} = \frac{2x-\cancel{3}-3x+\cancel{3}}{\cancel{2x}-3-\cancel{2x}+2}$$
$$= \frac{-x}{-1} = x$$

$$gof(x) = g(f(x))$$
$$= g\left(\frac{x-3}{x-2}\right)$$
$$= \frac{2\left(\frac{x-3}{x-2}\right) - 3}{\frac{x-3}{x-2} - 1}$$
$$= \frac{2x-\cancel{6}-3x+\cancel{6}}{\cancel{x}-3-\cancel{x}+2} = \frac{-x}{-1} = x$$

$\therefore$ $fog(x) = gof(x)$

$\Rightarrow$ f is invertible

$$f^{-1}(x) = \frac{2x-3}{x-1}$$